Using Pedagogical contexts to explore Mathematics: A Parallelogram Task in Teacher Education

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Mathematics and Teaching

- Mathematical Knowledge for Teaching
- Mathematical Territory
- Teachers’ Mathematical Work

Local (epsilon) neighborhood of the mathematics being taught

(Advanced) Mathematical horizon

(Curricular) Mathematical horizon

Teacher
Pedagogical Content Knowledge is primarily described as some melding of Mathematical and Pedagogical Knowledge - a subject-specific pedagogy.
Integrated Knowledge

- Mathematical/Pedagogical knowledge that is meaningfully integrated:
- How does teacher education promote integration of mathematical and pedagogical knowledge?
- How does teacher education elicit the mathematical and pedagogical work of teaching?
Tasks in Teacher Education

- According to Hiebert & Wearne (1993), “what students learn is largely defined by the tasks they are given” - which applies to prospective teachers as well.

- Tasks become the mediating tool for teachers’ learning; the quality of instruction and the success of the tasks depends on whether they unfold in ways that allow prospective teachers to learn through them (e.g., Kilpatrick et al, 2001).
Prominent Perspective

There seems to be a prominent perspective on tasks in teacher education designed that promote developing such integrated mathematical/pedagogical knowledge:
Tasks in teacher education may look like:

- One student says that the fraction (shaded blue) is 2/3. How might you as the teacher respond?

- Another student says it is 3/4, by drawing the following picture. Is the student correct? Explain. How would you justify this to the class?
Pedagogical Contexts

What might tasks in teacher education look like that develop integrated mathematical/pedagogical knowledge from the flipped perspective?
I. A Parallelogram Task
Recently, you introduced your class to the area formula for parallelograms, $A=bh$. You justified the formula by removing and relocating a triangle, as below.
Pedagogical Context

- Students’ conceptual understanding about area remains somewhat fragile and needs to be continually associated with enumerating square units.
- Students have difficulty transferring “remove-relocate” argument to unusually tall parallelograms (Wertheimer).
- Students have difficulty understanding that either length or width can serve as the base.
Discussion

- How might you approach preparing a lesson with these pedagogical considerations? Think about a task for students in a subsequent lesson.
  - What might be important?
  - What examples of parallelograms might you look at?
  - What concepts might you emphasize?
Bill's Parallelogram

Bill decides that he is going to model the Leaning Tower of Pisa with a parallelogram. However, he is having difficulty identifying dimensions to use. He would like every dimension (length, width, and both heights) to be integer values - in addition, he would like the height(s) to “split” the base(s) at an integer value.
Bill’s Parallelogram

- In other words, Bill wants $b_1$, $b_2$, $h_1$, $h_2$, $c$, and $d$ all to be integer values.

- Find values of $b_1$, $b_2$, $h_1$, and $h_2$ that meet this purpose - make sure they are also plausible for the Leaning Tower of Pisa. (What’s $\theta$?)
Bill’s Purpose

What might have been Bill’s purpose for wanting a parallelogram with these constraints?
Mathematical Work

- In general, what constraints on $b_1$, $b_2$, and $\theta$ (assume $\theta$ is the acute angle) result in such classes of parallelograms?

- $b_1$, $b_2$, $h_1$, $h_2$ are integers

- $b_1$, $b_2$, $h_1$, $h_2$, $c$, $d$ are integers
Mathematical Work

- \( \sin \theta = \frac{h_1}{b_2} = \frac{h_2}{b_1} \); Since these are integer values, \( \sin \theta \) is a rational number (between 0 and 1), which means there exist relatively prime \( m \) and \( n \) (with \( m < n \)) such that \( \sin \theta = \frac{m}{n} \).

- Since \( h_1 = b_2 \left( \frac{m}{n} \right) \) and \( h_2 = b_1 \left( \frac{m}{n} \right) \) are also integers, then \( n \) must divide both \( b_1 \) and \( b_2 \).

- Therefore, for \( b_1, b_2, h_1, h_2 \) to be integers, \( b_1 \) and \( b_2 \) must have a common divisor, \( n \), and the acute angle must be such that \( \sin \theta = \frac{m}{n} \) (for \( 0 < m < n \)).
Mathematical Work

- In order for c and d to also be integers, the value for n must be the hypotenuse length of a primitive Pythagorean triple (e.g., 5, 12, 13). And the value for m must be one of the other two values in the triple (e.g., 5, 12, 13).

- Thus, to construct a “Bill’s Parallelogram”:
  - Pick a value for n, the hypotenuse of a primitive Pythagorean triple; select a value for m, one of the other two values in the triple.
  - Select two multiples of n to be b1 and b2.
  - Select an angle so that \( \sin \theta = m/n \).

\[ \begin{array}{c}
  n = 13 \\
  m = 12 \\
  b_1 = 65 \\
  b_2 = 195
\end{array} \]
A Parallelogram Task

- This example used pedagogical contexts in teacher education tasks (such as the one described) as a means to explore mathematical considerations.

- Such tasks in teacher education help reveal the mathematical work of teaching, in ways that promote integrated mathematical/pedagogical knowledge.
II. Uses of Mathematical Knowledge in teaching
Uses of Knowledge

- **Decompressing**
  - Unpacking a topic's mathematical complexity in order to make it comprehensible

- **Trimming**
  - Removing complexity while maintaining mathematical integrity

- **Bridging**
  - Making connections across topics, assignments, representations, and domains

McCrory, et al., 2012
Uses of Knowledge

- Decompressing/Unpacking: Working with students' knowledge as it grows necessitates deconstructing teachers' own mathematical knowledge into less polished form, where elemental components are visible.
  - 1005 as one-hundred five
  - solving equations to allow interpretation of the results 0=0 or 0=3

- Complexities are in a local (micro-level) neighborhood of the context being taught
Uses of Knowledge

Trimming: Any concept can be taught in some intellectually honest way; carefully attending to ideas in a perhaps overly-polished form that removes or hides complexities with the intent of simplifying the concept:

- bad example: “multiplying makes bigger”
- slope of constant rate of change of linear functions in light of the ways it emerges as instantaneous in calculus

Complexities are in a broader (macro-level) neighborhood of the context being taught
Micro-level Trimming

- Bill's Parallelogram

- Mathematical issues related to area formulas, irrational lengths, piecing together partial squares, etc., are micro-level (local) complexities

- Intent was to remove/hide these complexities in order to further emphasize and make desired ideas clear, undistracted by aspects that may unnecessarily complicate
Macro-level Decompressing

- Square Perimeter
  - Perimeter as the sum of all sides is sufficient for polygons; but moving into circles complicates this idea in two ways: 1) multiplicative reasoning (not additive); 2) indirect measurement

- Dave developed the following task when discussing the perimeter of a square in light of this

How much longer is the perimeter of the square compared to the “middle” length pictured?
Macro-level Decompressing

- Square Perimeter

- Mathematical issues related to perimeter, multiplicative reasoning (not additive), indirect measurement, etc., are macro-level (local) complexities, e.g., circles

- Purposefully introduced (not removed/hidden) these complexities in order to prepare and unpack ideas for future developments of the concept, in ways intended to help students’ transition
Navigating Complexities

Conceptualizing Micro- and Macro-levels of Trimming and Decompressing in relation to the neighborhood of the mathematical complexities.

<table>
<thead>
<tr>
<th>Neighborhood of complexity</th>
<th>Response to complexity</th>
<th>Trimming (Removing complexity)</th>
<th>Decompressing (Unpacking/Highlighting complexity)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Micro-level</td>
<td>Local neighborhood</td>
<td>Micro-level Trimming</td>
<td>Micro-level Decompressing</td>
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<tr>
<td></td>
<td>(To make ideas comprehensible)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Macro-level</td>
<td>Distant neighborhood</td>
<td>Macro-level Trimming</td>
<td>Macro-level Decompressing</td>
</tr>
<tr>
<td></td>
<td>(To maintain integrity)</td>
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</tbody>
</table>
Implications

Part I: Tasks in teacher education that use pedagogical contexts to explore mathematical considerations promote integrated mathematical/pedagogical knowledge and reveal mathematical work of teaching.

Part II: 2x2 Framework for navigating mathematical complexities addresses both mathematical nature (local or distant neighborhood) and pedagogical responses (trimming or decompressing), and may be useful tool for teacher education.
Questions? Comments?

Thanks!

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