Exploring Teachers' Categorizations for and Conceptions of Combinatorial Problems

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K-12 Combinatorics

- K-12 Mathematics Education has had an increased emphasis on Probability and Statistics in the past two decades
- While **counting** is fundamental to calculating probability and understanding statistics, introductory combinatorics (e.g., permutations and combinations) are often tangential topics, only superficially discussed (if at all) in the K-12 curriculum.
- Is this due to rapid pace of curriculum? Teachers?

Expert Combinatorialists

- Counting is simple enough; yet counting problems span the spectrum of difficulty
- Expert combinatorialists have identified ways of organizing/categorizing different counting problems (e.g., Benjamin, 2009); yet doing so relies on a variety of modeling techniques for problems (Batanero, Navarro-Pelayo & Godino, 1997)

Combinatorial Organization

Selecting *k* objects from *n* distinct objects

	Ordered (permutations)	Unordered (combinations)
Without repetition	**Arrangements** $\frac{n!}{(n-k)!} = n\cdot(n-1)\cdot\ldots\cdot(n-k+1)$	**Subsets** $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
With repetition	**Sequences** $n^k = \underbrace{n\cdot n\cdot n\cdot\ldots\cdot n}_{k}$	**Multisubsets** $\left(\!\binom{n}{k}\!\right) = \binom{k+n-1}{n-1}$

Adapted from Benjamin, A.T. (2009, p. 10)

Novice Combinatorialists

- Experts may recognize the characteristics of combinatorial problems according to the 2x2 matrix distinctions, and understand how to model them accordingly
- Yet do those learning to think combinatorially make the same connections or distinctions?
- While many problems can be organized along the 2x2 matrix, modeling them in ways that fit those descriptions may be unnatural & difficult

A Problem

- How many numbers between 1 and 10,000 have the sum of their digits equal to 9?

Modeling Difficulties

- Technically, this is a multisubset problem (*unordered*, with *repetition*). However, to **model** this as a multisubset requires using letters to represent the four *distinct* digits, like T=tens digit. One answer in this model may be: TTTHTOTOO (153)
- So if the expert categorization can be difficult to utilize when solving, how can we better capture a learners' development of combinatorial thinking?

Actor-Oriented Transfer

- Identifying ways to apply knowledge from previously learned problems to another context is generally known as *transfer*
- Actor-oriented transfer (AOT), characterized by Lobato (2003), shifts the perspective regarding transfer from an expert's view to a learner's vantage point.

Actor-Oriented Transfer

- Lockwood (2011) argues that AOT may be particularly useful for combinatorial thinking.
- AOT pays attention to the ways novices draw on their knowledge to solve problems in another context.
- Given the structural and modeling aspects of combinatorics, documenting examples of AOT from novices, should help understand how combinatorial thinking develops.

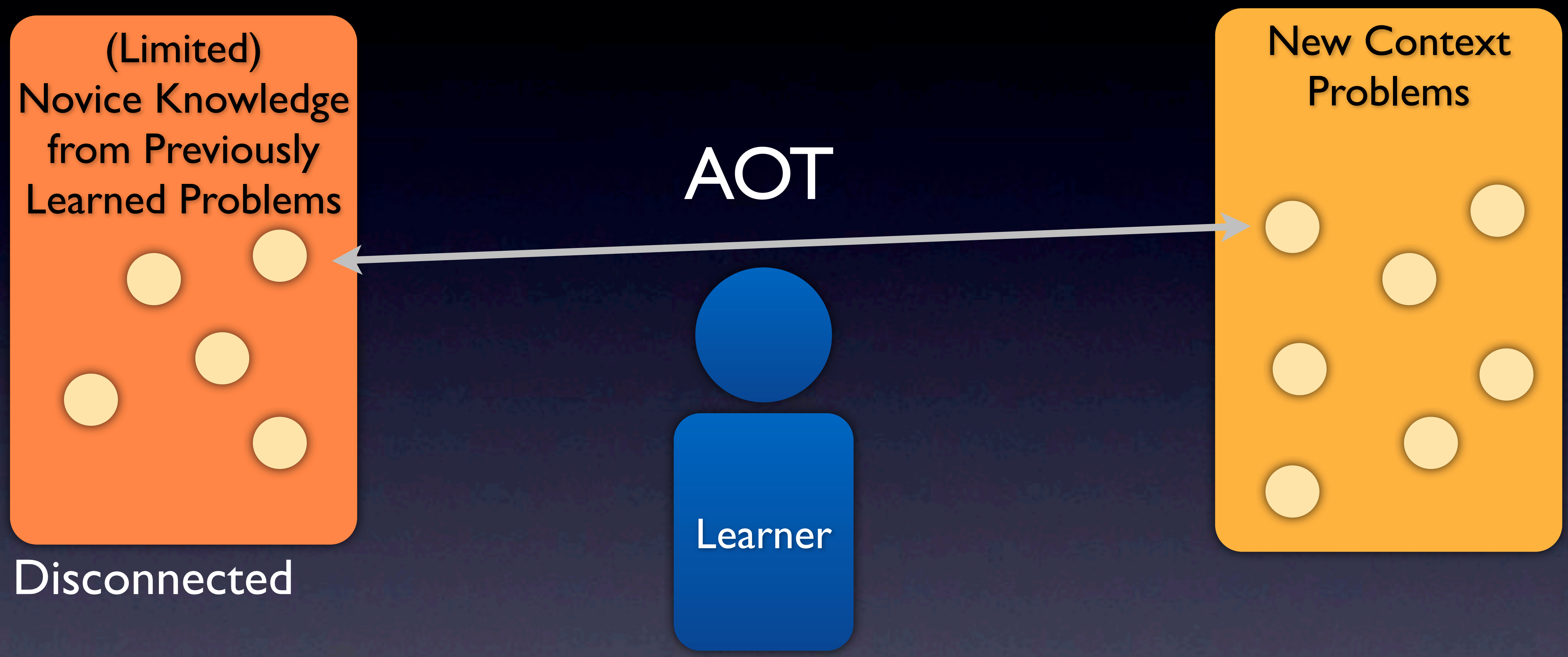
Research Question

- How do middle and secondary mathematics teachers, who are also novice combinatorialists, categorize and conceptualize different combinatorial problems?

Framework


(Limited)
Novice Knowledge
from Previously
Learned Problems
Disconnected
New Context
Problems

Framework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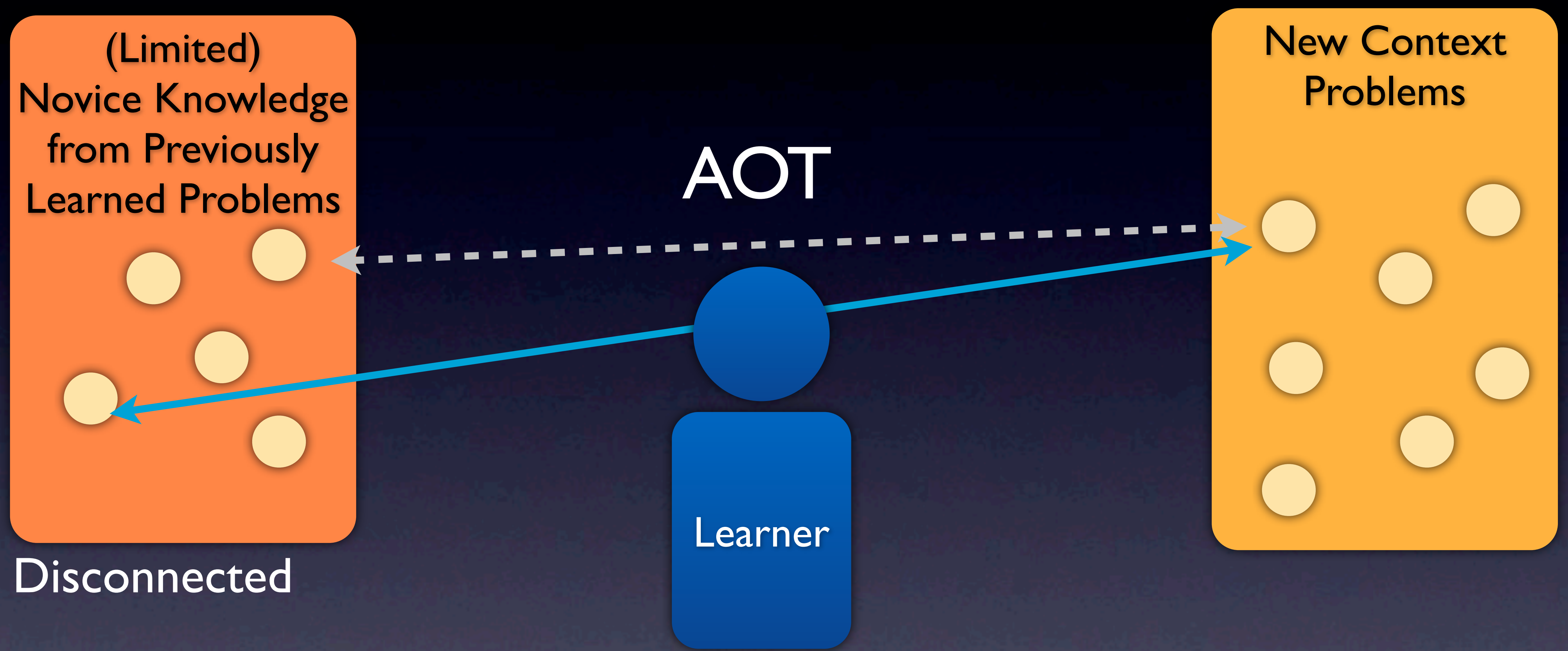

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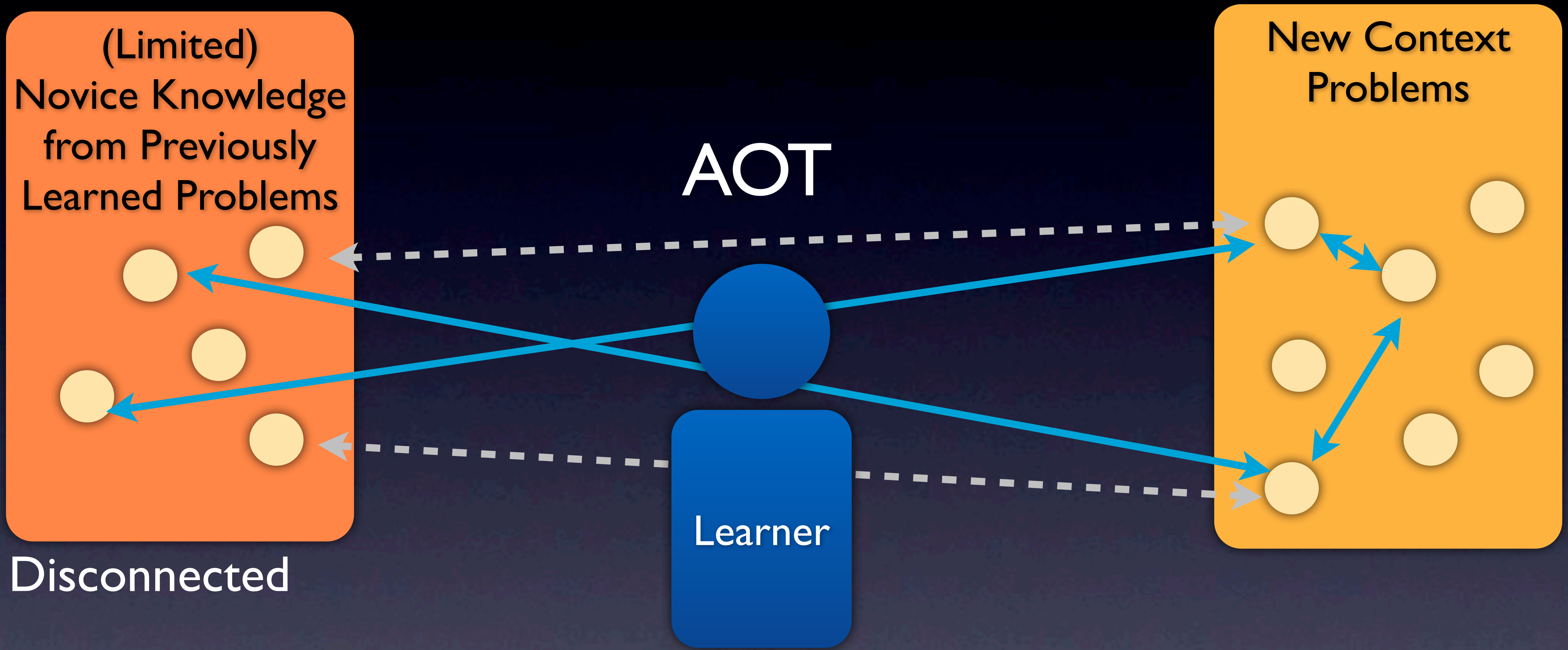

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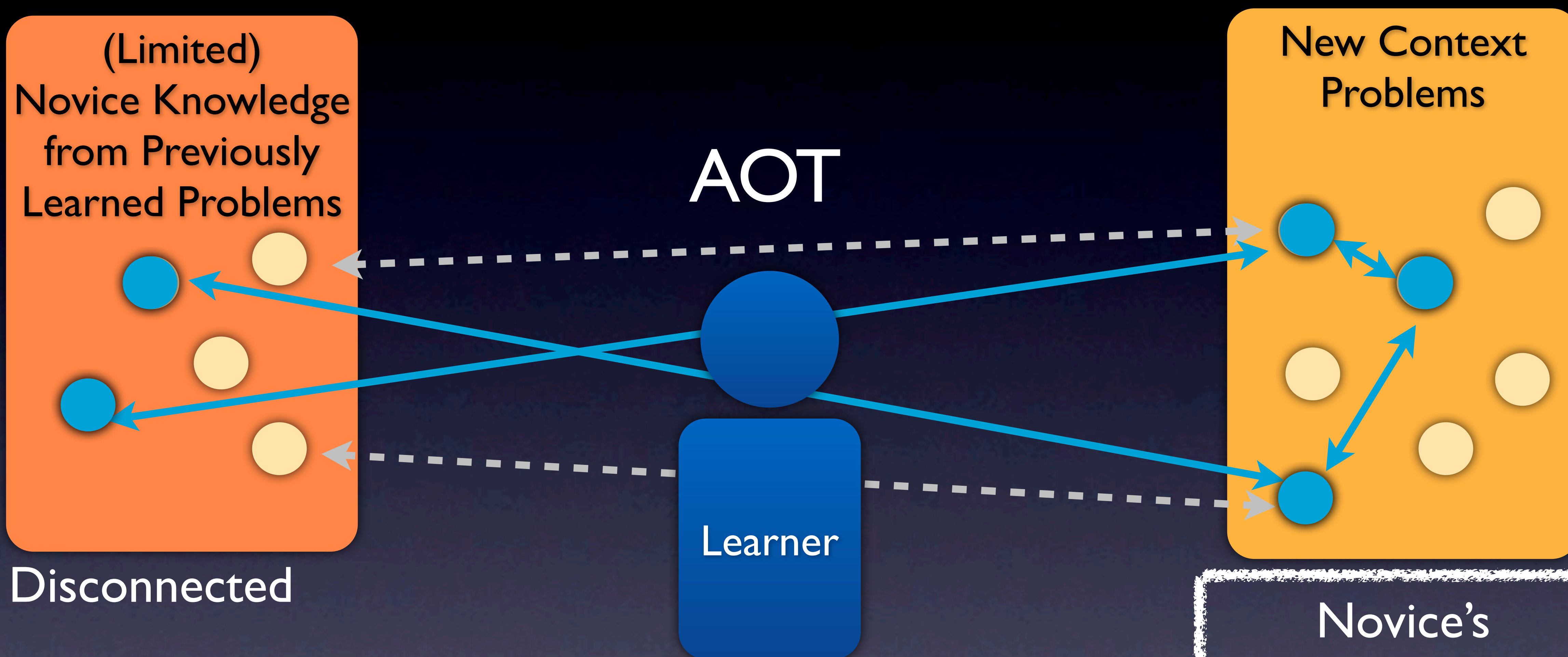

Documenting instances of PRODUCTIVE and NON-PRODUCTIVE AOT informs the development of combinatorial thinking → Novice's organization and conception of combinatorial problems

Methodology

- Two focus groups (e.g., Berg & Lune, 2012) conducted with practicing middle and secondary mathematics teachers (N=3; N=4)
  - *None had taken a combinatorics/discrete course*
- In conjunction with graduate mathematics education course (Teaching Probability & Statistics)

Methodology

- Introduction (~90 minutes) in course consisted of:
  - Introduction and instruction on: the addition principle; multiplication principle; factorial notation; and $\binom{n}{k}$ notation
  - Approaches and solutions to six combinatorics problems (relatively common examples, spanning all four problem types)
    - NOTE: various strategies for solving problems were introduced, but no structural characteristics were mentioned (e.g., order matters, etc.)

Methodology

- After introduction, 7 study participants were randomly assigned to one of two focus groups (~120 minutes)
  - They were asked to work on an assortment of 12 problems, ranging in type and complexity

    "Answer each of the problems and organize them into 'groups' of problems that have similar methods for solving. For each group of problems, provide a brief description of how and why the problems in that group are similar."
- Researcher took field notes about important comments or connections made by participants, at times asking questions to uncover their thinking (*next time, SmartPens)

Overall Findings

- Permutation Problems

Table 3: The groups' categories and descriptions for *permutation* problems

Type	Problems	Group 1		Group 2	
		Problems	Description	Problems	Description
Arrangements (Ordered, without repetition)	Problems States Netflix Plane Routes	States (Texas) Netflix Plane Routes	-Each thing can only be in one place at a time (no repeats within set) -Order matters	States Netflix Plane Routes *M/F Committees*	-The order of choices matters. -Choices cannot be repeated.
		Problems	Description	Problems	Description
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Overall Findings | Preferred Vantage Point | Approaching Multisubsets

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Subsets (Unordered, without repetition)	Handshakes Supreme Court Voting MC Exams2 M/F Committees	[missing] Supreme Court Voting [did not do] [did not do] *Hot Dogs*	-How many different positions each element can occupy (9 choose 6)	Handshakes Supreme Court Voting MC Exams2 [in Arrangement]	-Take the groups and choose a certain number. Using the "group" choose "number" gets rid of the duplicates. The duplicates exist because order does not matter.
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- Group 1 also had an additional category with Vowels & MC Exams2
- Their characterization was essentially that, for these problems, you have to consider Cases (which is true)

Preferred Vantage Point

- How many ways can you distribute a $1, $2, $5, $10, and $20 gift card 8 friends. (i.e., you have 5 gift cards to distribute)

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- How many ways can you distribute a \$1, \$2, \$5, \$10, and \$20 gift card 8 friends. (i.e., you have 5 gift cards to distribute)
- Both groups began by drawing 8 slots (each person)

- Their attempts included (8 choose 5)
  - “but then someone could get more than one”
- and 5^8
  - “but then the last person wouldn’t have 5 choices”

Overall Findings | Preferred Vantage Point | Approaching Multisubsets

Preferred Vantage Point

- Trying to count *which person* receives which gift card(s) causes **modeling difficulties**: each person could have anywhere from 0 to 5 gift cards, and sequential models (i.e., eight slots) make the result for subsequent persons dependent on previous ones.
- To solve it from this perspective would require accounting for each of the *7 distinct integer partitions of 5*, and then distributing the gift cards according to these possible partitions, which becomes quite complex.

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- It was not until the participants shifted from the *perspective of the people*, who are receiving gift cards, to the *perspective of the gift cards*, which are being distributed, that progress was made.

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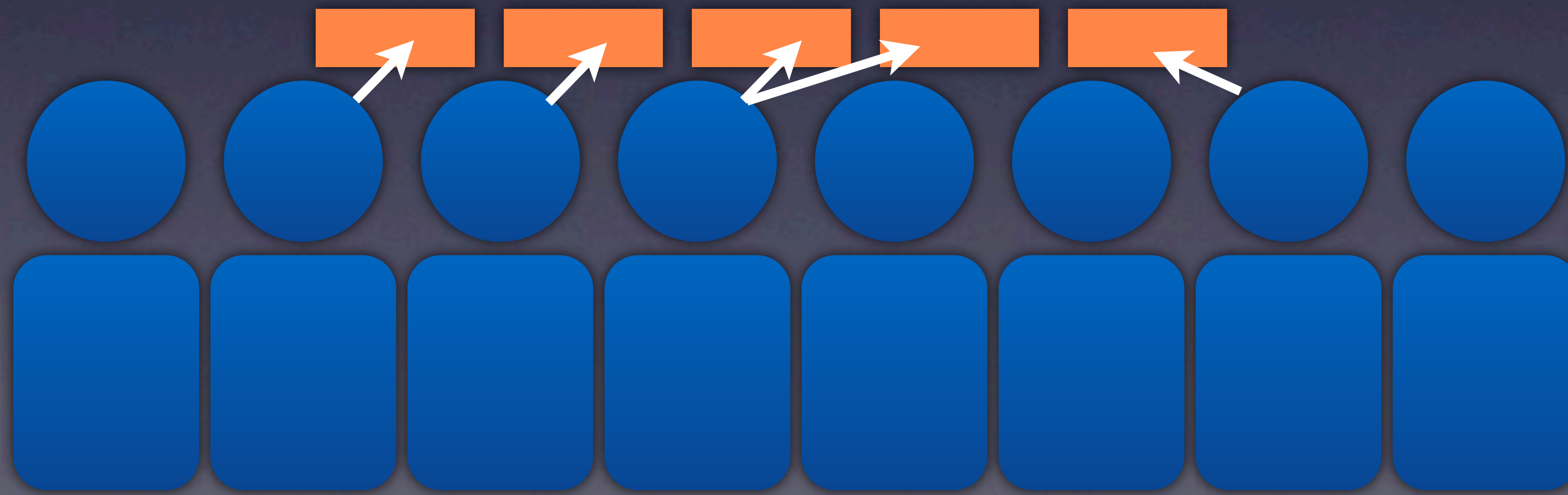

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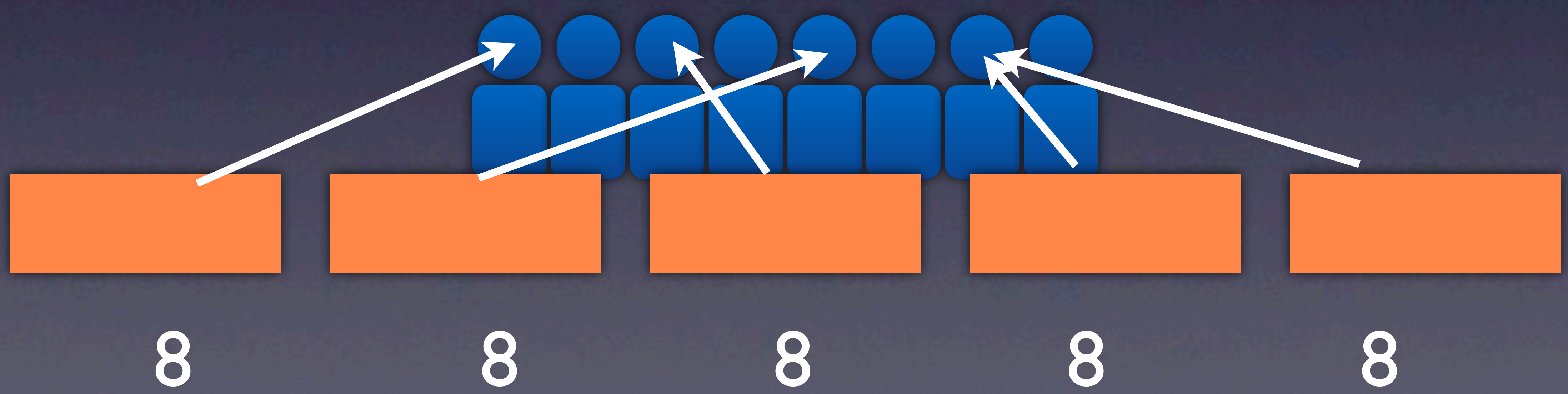

Preferred Vantage Point

- Combinatorics problems frequently have two perspectives from which to model (e.g., *n* objects and *k* objects)
  - functions: input → output (other direction can be difficult)
- It may be that students bring a particular *preferred* perspective to some problems (that is, for whatever reason hard to overcome/switch), that can cause difficulties for them in solving

Overall Findings | Preferred Vantage Point | Approaching Multisubsets

Approaching Multisubsets

- How many ways are there to make a pizza with **1 topping**, if the choices for toppings are: Pepperoni, Olives, Sausage, Ham, Mushrooms, and Anchovies?
- **2** toppings (*double toppings allowed*)?
- **3** toppings (*double and triple toppings allowed*)?
- **4** toppings ?

Approaching Multisubsets

- The group's approach to solving the 2 topping pizza problem was:

  2 toppings: $\binom{6}{2}$ +

  1 (double) topping: $\binom{6}{1}$

- Generally, however, these problems are Multisubset problems
  - Select **4 toppings** from 6 distinct toppings (can repeat toppings): PPOS & PSOP are same (unordered)

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Pepperoni Olives Sausage Ham Mush Anchovie

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P O H H

Approaching Multisubsets

- Interestingly, their approach to other Multisubset problems mirrored their work on the pizza problem.
    - For the 2 topping pizza problem, they split it into two simpler cases: 1) two *different* toppings (6 choose 2); and 2) one double topping (6 choose 1)
- We return to their work on the Summed Digits Problem to illustrate their similar approach

Approaching Multisubsets

- How many numbers between 1 and 10,000 have the sum of their digits equal to 9?
- Case 1: the "9" is contained within one digit. There are (4 choose 1) of those: 9000, 900, 90, 9.
- Case 2: the "9" is split between two digits. There are (4 choose 2) pairs of digits, and each of those have 8 (ordered) possibilities: (1,8), (2,7), (3,6), (4,5), (5,4), (6,3), (7,2), (8,1).
  - So picking the Thousands and Tens digits: 1080, 2070, 3060, 4050, 5040, 6030, 7020, 8010

Approaching Multisubsets

- Case 3: the “9” is split between three digits. There are (4 choose 3) pairs of digits. There are then 7 partitions of 9: (7, 1, 1), (6, 2, 1), (5, 2, 2), (5, 3, 1), (4, 3, 2), (4, 4, 1), (3, 3, 3); however, the number of ways to “order” the partitions are different (e.g., there are 3 ways to order (7,1,1), 6 ways to order (5,3,1) and only 1 way to order (3, 3, 3)). This makes a total of 28 ways to “order” these.
- Case 4: similar difficulties...four digits

Approaching Multisubsets

- How many numbers between 1 and 10,000 have the sum of their digits equal to 9?

Case 1 (split among 1 digit): $\binom{4}{1}$

Case 2 (split among 2 digits): $\binom{4}{2} \cdot 8$

Case 3 (split among 3 digits): $\binom{4}{3} \cdot 28$

Case 4 (split among 4 digits): $\binom{4}{4} \cdot ?$

Approaching Multisubsets

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Case 1 (split among 1 digit):	$\binom{4}{1}$	$\binom{4}{1}\binom{8}{8}$
Case 2 (split among 2 digits):	$\binom{4}{2}\cdot 8$	$\binom{4}{2}\binom{8}{7}$
Case 3 (split among 3 digits):	$\binom{4}{3}\cdot 28$	$\binom{4}{3}\binom{8}{6}$
Case 4 (split among 4 digits):	$\binom{4}{4}\cdot ?$	$\binom{4}{4}\binom{8}{5}$

Approaching Multisubsets

- How many numbers between 1 and 10,000 have the sum of their digits equal to 9?

Case 1 (split among 1 digit):	$\binom{4}{1}$	$\binom{4}{1}\binom{8}{8}$
Case 2 (split among 2 digits):	$\binom{4}{2}\cdot 8$	$\binom{4}{2}\binom{8}{7}$
Case 3 (split among 3 digits):	$\binom{4}{3}\cdot 28$	$\binom{4}{3}\binom{8}{6}$
Case 4 (split among 4 digits):	$\binom{4}{4}\cdot ?$	$\binom{4}{4}\binom{8}{5}$

Case 1 (1 double topping):	$\binom{6}{1}\binom{1}{1}$
Case 2 (2 toppings):	$\binom{6}{2}\binom{1}{0}$

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In general, their approach to Multisubset problems, selecting *k* from *n* distinct objects, can be generalized as: $\sum_{i=0}^{k-1}\binom{n}{k-i}\binom{k-1}{i}$

Approaching Multisubsets

- Technically, this is a Multisubset (with repetition; unordered), where we are selecting 9 (*k*) from 4 (*n*) distinct objects. The "distinct" objects are place values (O=ones; T=tens; H=hundreds; M=Thousands).
- In this case, the "order" does not matter, each number is represented by the amount of O, T, H, and M's that are selected:

TTTMTTMTT = TTTTTTTMM = 2,070

Approaching Multisubsets

- How many numbers between 1 and 10,000 have the sum of their digits equal to 9?
- Stars & Bars is perhaps the best Model

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Overall Findings | Preferred Vantage Point | Approaching Multisubsets

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| | | | | | | | | ★ ★ ★

Thousands Hundreds Tens Ones

| | ★ ★ | | | | | | | ★ 2,070

Th Th T T T T T T T

Total: 12 choose 3

Conclusions

- Generally, teachers had more success *categorizing* and *characterizing* Permutation (ordered) problems than Combination (unordered) problems
- A *preferred vantage point* may impact/hinder productive approaches to solving combinatorics problems; learning combinatorial reasoning may involve being able to *shift* perspectives
- Splitting Multisubset problems into cases (with more natural models) was students' tendency; this may suggest ways to help students transition and scaffold their thinking to more helpful (and computationally easier) approaches

Thank You!

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