Partition and Iteration in Algebra: Intuition with Linearity

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If a basketball team scores 22 points in the first half, how many do they score in the second [Figure 1]? If one state gets 2 senators, how many do other states get? If someone's salary is $2,000 in January, how much is received in other months? While there are many different plausible answers these questions, the most common answer is equal – 22 points in one half means 22 in the second; 2 senators for one state means 2 for others; $24,000 in salary should be split equally into $2,000 a month. Equality is perhaps the easiest thing to assume, the simplest method of prediction. If a state with 1 million people has five congressmen, then a state with 2 million people should have ten. The notion of proportionality has deep roots within humanity. This, in essence, is the assumption of linearity.

Figure 1: Photo by: Richard Kligman (Flickr) http://www.flickr.com/photos/rkligman/3401091109/

There are many things in the real world that have, or can be modeled by, linear relationships – currency exchange, translating between units of measure, speed and distance, gain/loss over time, and some geometric patterns to name a few. The notion of linearity boils down to the supposition that there is some *constant* rate of change, or slope, that can relate two variables. Using an image of buckets that we will continue to refer to – whether the bucket is a half of a basketball game, a state, or a month – linearity presumes that every "bucket" should hold the same amount of stuff. Rate of change and slope are often the vocabulary used to help students understand linear relationships. Yet in a recent article connecting research to teaching, [Teuscher and Reys, 2010] give evidence that students misunderstand these two ideas and have difficulties conceptualizing and connecting them. In this article we will explore the concepts and vocabulary of partition and iteration as a vehicle to make use of students' intuition about slope and linearity, using tables as a primary organizational tool.

Partition and Iteration

Figure 2: http://www.freeusandworldmaps.com/

Partitions have deep significance in theoretical mathematics. We may seek to partition a set, or look for the number of ways to partition a positive integer in number theory. A natural image of a partition is a map [Figure 2], where a country is divided into states by boundary lines so that each place belongs to one and only one state, and all the states together make up the whole country. While partitions have very specific and sophisticated mathematics notions connected with them, students are often first introduced to the idea in elementary school. A partition involves splitting something into some arrangement of disjoint parts, and this is first introduced with division and fractions – splitting into *equal* parts. Students learn to take 48 candies and partition/divide them evenly among 8 friends; or to represent 2 things split into 3 equal parts as the fraction 2/3. As a matter of fact, people often partition, divide, or split up equally since equality is so natural to our thought process – earning $50 for 5 hours of babysitting means $10/hr as opposed to some other unconventional method of payment.

Iteration is another notion that has deep roots in very significant mathematics; dynamical systems, fractals, and computations of irrational numbers are a few ideas that involve iteration. Yet very simply, iteration is the process of repeating something, often infinitely or cyclically. And while not always obvious to students, they are first acquainted with this in elementary school when learning to count by 5's or 6's (skip counting), which leads to the operation of

Figure 3: Piece of an Infinite Hole, by Jen Stark
Photo by: Harlan Erskine
http://www.harlanerskine.com/blog/category/tear-sheet

multiplication. It is also inherently present in understanding fractions and counting with fractions – one-eighth, for example, is the amount such that 8 one-eighths put together form a whole.

In some sense, partitioning and iterating are almost opposing concepts: in one you take a whole and split it into smaller parts; in the other you take the smaller parts and copy/repeat to re-create the whole [Figure 3]. So while both partition and iteration play much deeper and more significant roles in mathematics, students are often introduced to them very early on with multiplication, division and fractions [Siebert & Gaskin 2006]. And while this vocabulary may not be explicitly used in elementary school, making use of students' familiarity with previous ideas when developing the notions of linearity, slope, and rate of change, serves to reinforce students' understanding. As described by Hoven and Garelick [2007], regular use of a bar model to represent fraction questions in Singapore's mathematics education program enriched students' learning through its consistent use throughout a vertically aligned curriculum.

Partitioning and Iterating in the Algebra Classroom

Building out from the basic notion of partition and iteration, we will look at how these processes can help students' formation of the ideas of linearity, slope, and rate of change in the context of a few typical problems. The term slope can be very intuitive when discussing linearity, but it has its unfortunate shortcomings, in part stemming from the imprecise use of the word in the English language (i.e. discussing a "steep slope" or a "curvy slope" might be acceptable when describing a mountain, but is too general or is a paradox in mathematics). Its mathematical meaning, in contrast, limits how slope may be correctly referenced. The strict definition that quantifies it into a precise number distinguishes clearly between slopes but can also be a source of confusion for students (i.e. a slope of 4 and 5 are very different mathematically, but both might be described as "steep"). Rate of change, too, is incredibly useful for discussing linearity, particularly in problems whose variables can be discussed in terms of sensible rates (i.e. miles per hour, meters per foot, etc.), but can lack relevance in more peculiar connections that might be interpreted as quarters per penny, or pencils per eraser. While both slope and rate of change will prove useful for conversations about linearity, the progression of problems discussed is geared to build up the important concepts needed to understand linearity making use of students connections to the ideas of partition and iteration, which is beneficial for its intuitive relation to elementary topics among other things.

***Conversion* – Implications for a Constant Slope**

A common place to begin discussing linearity, because of the proportional relationship involved, is through the idea of conversion – converting between currencies or various measures. The following problem might be given:

One Euro is equal to $1.47 in U.S. dollars. How much are 2, 3, 4, 5, 6 or 50 Euros worth in American dollars?

€	$
1	
2	
3	
4	
5	
6	

Figure 4

Making use of a table [Figure 4] as a way of organizing responses, students can simply add the constant rate, $1.47, each time. This results in the values $2.94, $4.41, $5.88, $7.35, and $8.82. To find the value of 50 Euros, students might add $1.47 fifty times, or better yet, might multiply because they learned at some point that repeated addition is multiplication. Regardless, the process of iteration clearly displayed in the table, e.g. repeatedly adding the constant rate, is essential to developing a sense of linearity – that students effectively keep adding the same number over and over.

This repeated addition of a constant difference is a useful introduction to the next question that allows students to wrestle with how to make use of the iterative process:

A package weighs 6 kilograms, or 13.2 lbs. What does a 1-kilogram package weigh in lbs.?

kgs	lbs
0	0
1	
2	
3	
4	
5	
6	13.2

Figure 5

While many students might approach this as a simple division or proportion problem, an important connection relating partition and iteration can be established here that is useful in understanding linearity. Using a similar table strategy [Figure 5], students will need to find some constant number added 6 times that will increase to 13.2 pounds. From experiences in class discussions, the question quickly becomes, "How do you split 13.2 pounds into 6 equal increments?" – e.g. how do you partition, divide, or split up 13.2 into 6 equal partitions? This simple division problem yields 2.2 as an answer. The advantage of understanding both the partitioning and iterating processes is the ability to verify correct and incorrect answers. By adding 2.2 lbs each time into the table like in the previous problem, at six kilograms the answer would be 13.2 pounds, verifying that the rate is correct. [See Figure 6.]

kgs	lbs
0	0
1	2.2
2	4.4
3	6.6
4	8.8
5	11.0
6	13.2

Figure 6

***Driving to New York City* – Finding an Initial Value**

Partitioning equally ultimately becomes the process in which students understand how to identify the constant slope or rate of change. Iterating ultimately becomes the process in which students understand how to use slope or rate to find other values. Both of these ideas become transparent looking at tables of information. Take the following question:

> *After driving four hours a car is 251 miles away from NYC and after driving seven hours it is 35 miles away from NYC. Assuming a constant speed, how far away did the car begin its trip?*

time	distance
0	
1	
2	
3	
4	251
5	
6	
7	35

Figure 7

Using tables to organize, students can paraphrase this question as: Fill in the following table assuming a linear relationship [Figure 7]. In particular, this prepares students for working backwards to find an initial value, in this case $t = 0$. Once the information is organized, students can see that the car has traveled 216 miles in the last 3 hours. To partition the distances and time spent traveling is to calculate that 72 miles is the constant rate that would equally divide the total number of miles. Iterating then becomes subtracting 72 miles each time, which fills in the numbers corresponding to 5 and 6 hours. Continuing this pattern, or adding 72 miles up the table, results in finding that the car started 539 miles away from NYC [Figure 8]. Thinking in terms of the "bucket" analogy [Figure 9], each bucket representing one hour, partitioning equally focuses on what *one* bucket contains, 72 miles, while iterating makes sense of the *many* buckets.

time	distance
0	539
1	467
2	395
3	323
4	251
5	179
6	107
7	35

Figure 8

partition iteration

Figure 9

***A Baby's Weight* – Linear Assumptions**

In addition to allowing for a deeper discussion of slope and rate of change, the processes of partitioning and iterating can also be used to expose the assumptions of linearity, and to quickly compute other values in a linear model. Equal partitions expose the constant rate assumption of linearity; iteration in the form of multiplication allows for efficient calculating. To demonstrate, we adapt a problem from the *Discovering Algebra* textbook (p. 163):

A baby weighs 6.8 lbs at birth and at 6 months the baby weighs 15.8 lbs. How much does the baby weigh at 1, 2, 3, 4 and 5 months? How much will the baby weigh at 2 years (24 months)?

age(mo)	weight
0	6.8
1	8.3
2	9.8
3	11.3
4	12.8
5	14.3
6	15.8

Figure 9

Intentionally, this question is stated in a manner meant to draw out limitations of linearity, making it impossible to solve without further information. Posing the first question is meant to get at the assumption of equal rates. Some students might respond that the baby must gain 1.5 lbs each month (dividing to find a constant rate of change). Others might claim that this is not necessarily the case since the baby might gain 1 lb the first month, almost 2 lbs the fourth month, etc. [Figures 9 and 10]. Leaving room in the question for this discrepancy allows for a conversation about the benefits and limitations of linear models, which *assume* a constant rate. While perfectly linear models might not be totally accurate, nonetheless they are often very good estimates over short periods of time for many relationships (local linearity). Facilitating class discussions like this helps expose the underlying assumptions of linearity, allowing students to grasp slope, an equal partition, in a rich way.

age(mo)	weight
0	6.8
1	7.8
2	8.8
3	9.9
4	11.8
5	13.3
6	15.8

Figure 10

The second question poses a challenge to efficiently justifying and calculating other values. If we assume linearity for the baby's weight gain (9 lbs in 6 months), and only if we do so, then the baby will gain 1.5 lbs each month. For the purposes of this discussion here we will continue using this linear framework, though an interesting discussion about linearity can be had from students collecting actual data on a baby's development of weight over time to see how good the linear assumptions are, and at what age that linear model might stop being useful (local linearity). Assuming linearity, the answer to the question can be found by adding 1.5 lbs each month on a table. As students begin filling in this constant increase each month, experiences with middle and high school students have led to them quickly looking for more efficient ways of calculating – the shortcut of multiplication, repeated addition. For 18 months the baby will grow 1.5 lbs each month, so 18 times 1.5 will give the total weight gain for these 18 months, which is 27 lbs. Adding this 27 lb increase to the weight at 6 months gives a total weight of 36.8 lbs for the baby. Using tables and the idea of iteration, which turns into multiplication in the case of linear relationships, students can begin to generate and justify more efficient ways of answering specific questions involving linearity.

***Losing Weight* – Writing Equations**

So far we have developed basic ideas of slope – that it can be found by equally partitioning and that it can be used in an iterative process – in problems dealing with *conversion*; we have developed ways of finding an initial value through division and iteration in *driving to New York City*; and we have exposed linear assumptions by partitioning and have developed more efficient ways of calculating – multiplication – by understanding iterating in *a baby's weight*. All these pieces come together for the process of writing linear equations. It will first be important that teachers justify the starting value, or $x = 0$, is *the* important starting point for writing equations of lines. Whether teachers do this by talking about how knowing the weight of the baby at birth $(t = 0)$ is important for understanding the weight over time, or whether teachers are more formal and show that *only* if you start at $x = 0$ will the number of times you repeatedly add the value of the slope be the independent variable, x. Regardless, once a case is made that you need the starting value corresponding to $x = 0$ to write the equation of a line, students can use the notions of partition and iteration to efficiently find and justify a linear equation.

Betty has been on a weight loss program for the last 12 months. Seven months after starting the program her weight was 142 lbs, and today (at 12 months) she weighs 129 lbs. Assuming linearity, write a linear equation that tracks her weight over time.

Having now understood linearity assumes that Betty loses the same amount each month, students can begin by trying to understand how much she is losing by focusing on *one* "bucket," or month. Betty has lost 13 lbs in 5 months, which means she is losing 2.6 lbs per month. Once students know that the starting value, or $t = 0$ point is important, multiplication becomes an efficient way to calculate it, similar to the problem *driving to New York City*. Needing to go back 7 months, at 2.6 lbs per month (seven 2.6 lb "buckets," or months), means a total of 18.2 pounds. So her original weight is $142 + 18.2 = 160.2$ lbs. Students can then use this starting value as the beginning of writing the equation, $w = 160.2 - 2.6t$. Each of these numbers can be understood in relation to the student's comprehension of rate of change. Making use of equal partitions helps students grasp what linearity assumes and why 2.6 is the rate of change; using multiplication as iteration helps justify why 160.2 lbs. is her initial weight, as well as why the equation would include multiplying 2.6 times t months.

Capitalizing on the elementary processes of partition and iteration, finding slope can easily be generalized into finding the gain/loss of the y values, and dividing it over the difference of the x values, i.e. the familiar $\frac{y_2-y_1}{x_2-x_1}$. [See Figure 11.] The process of iteration allows us to find what corresponds to $x = 0$ given any point (x_1, y_1) by multiplying the slope value "x" times and subtracting it from the "y" value, i.e. $y_1 - mx_1$. This value becomes the starting point for writing the equation of a line, which could then be written as $y = (y_1 - mx_1) + mx$. As this form for the equation of a line is very unconventional, I do not advocate this generalized form over others already established. Rather, it simply suggests that the processes for using partitions and iterations can lead to generalized results that are meaningful to students.

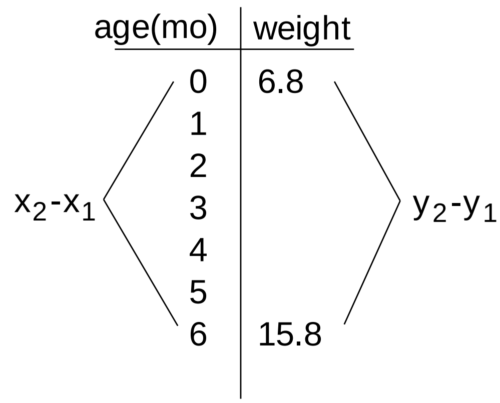

Figure 11

Conclusion

The primary purpose of this article is to look at how the notions of partition and iteration can be helpful processes for understanding slope and constant rates of change. Throughout, tables were the main visual tool employed to understand and conceptualize these two processes as they relate to linear relationships. Exploiting tables as a helpful organizational tool allows students the opportunity to make use of their intuitive sense of equality and basic operations like multiplication and division. Yet despite some of the benefits, the notions of partition and iteration, and particularly the "bucket" analogy used, have limitations as well. Students are slow to begin discussing fractions of a "bucket", and justifying how much would be in one-third of a "bucket", or one-third of an iteration – though this can be done. And a deep comprehension of linearity requires that students be able to understand rational and real values for the independent variable, and not just integer values. Yet in conjunction with, and not at the exclusion of, more common terms like slope and rate of change, these two vocabulary words are intuitive and useful. In ways that distinguish them from slope and rate of change, partition and iteration not only encourage ideas of equal distributions and repeatedly adding some constant value, but they also expose assumptions of linearity that are helpful for students' further study in mathematics. Their elementary connection to the vocabulary of fractions also poses a cyclical benefit since slope is often discussed in terms of fractions, making students' understanding of fractions reinforce the notion of slope and the discussion of slope reinforce a better understanding of fractions. The disconnect graphically with a slope, say, of ¾, is often evidenced in students' inability to vocalize it as anything other than "up 3 and over 4." Using partition and iteration, ¾ becomes not just two separate integers, but also one number – the amount the "y" value increases for each "bucket," or "x" value [Figure 12]. Linear relationships create a natural setting for students to better understand fractions and their worth, and illustrate some of the benefits that the processes of partition and iteration hold to exploit an intuitive understanding of the important concepts of rate of change, slope, linearity and linear equations.

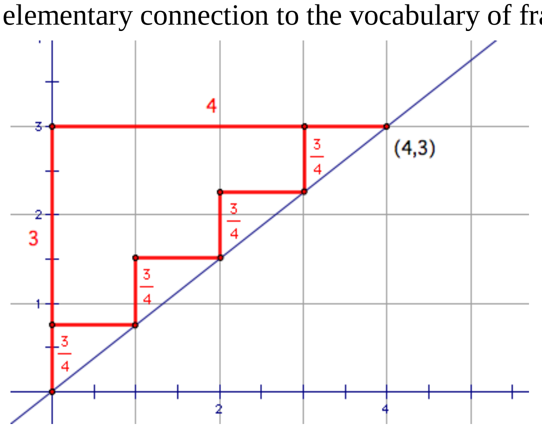

Figure 12

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