Using Pedagogical contexts to explore Mathematics: A Parallelogram Task in Teacher Education

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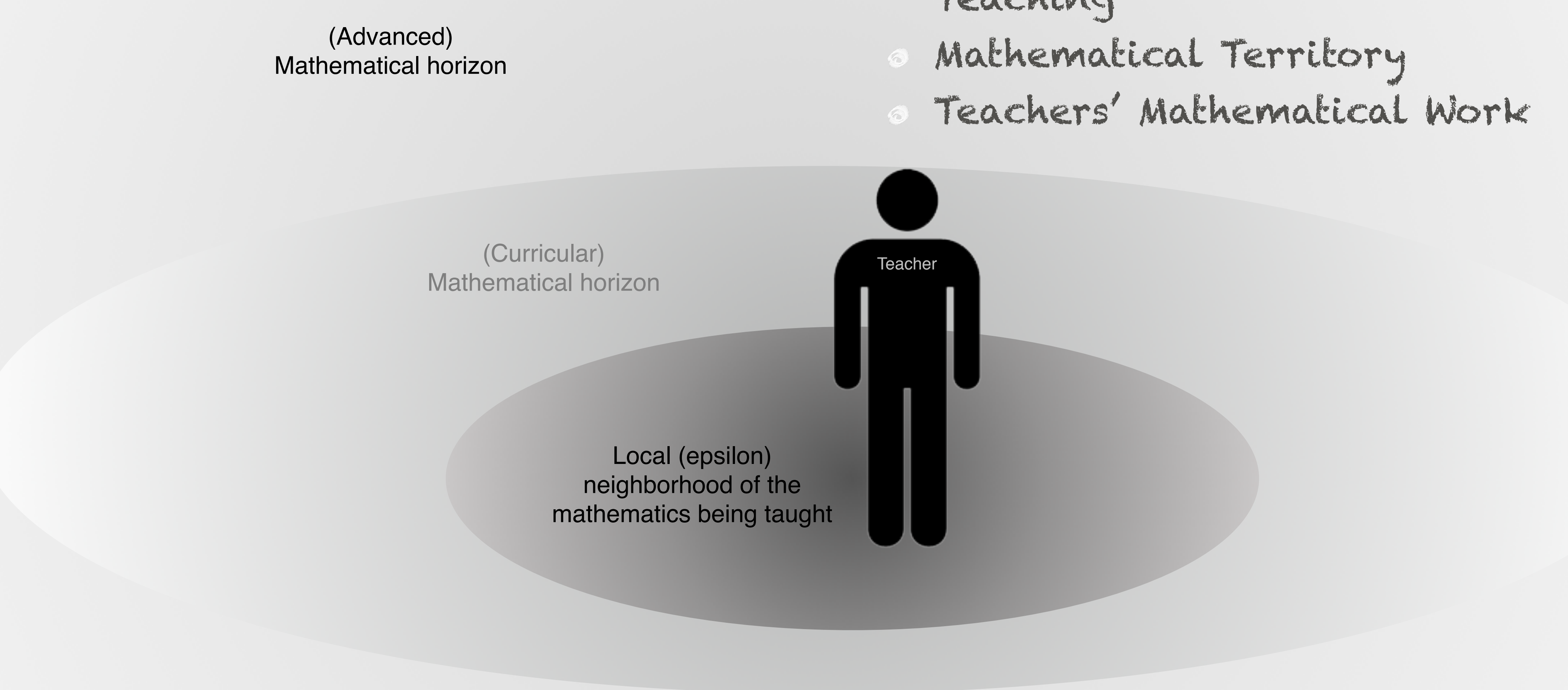
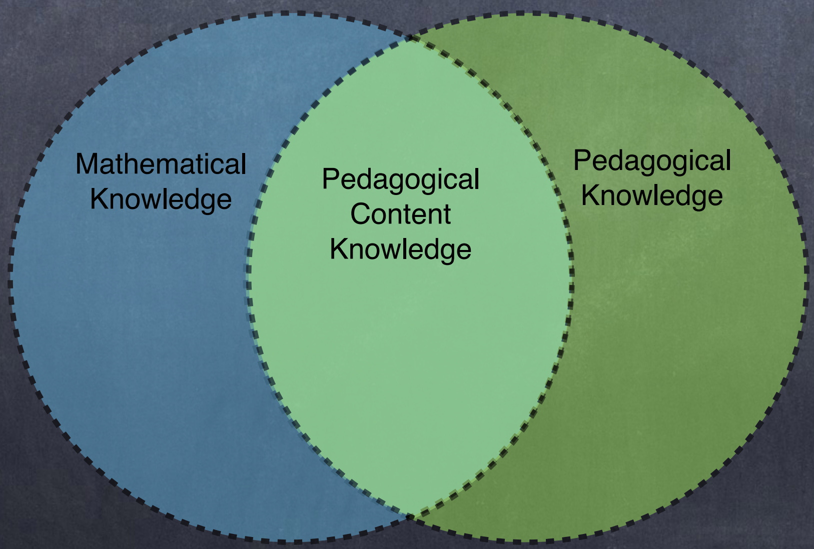
Mathematics and Teaching

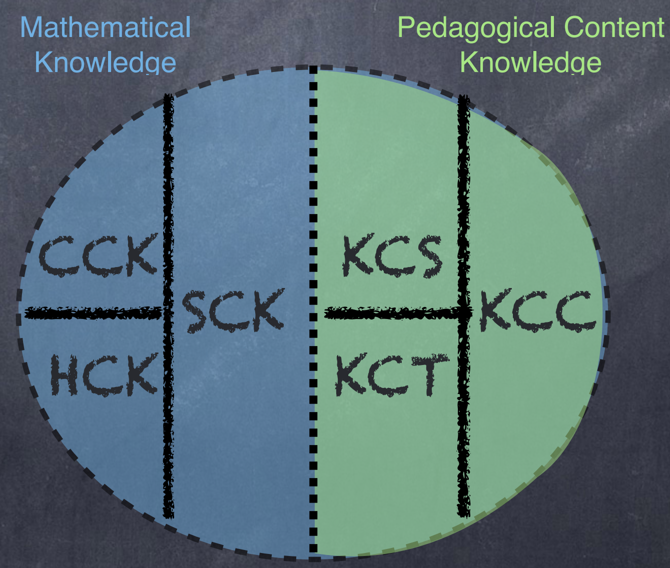
- Mathematical Knowledge for Teaching
- Mathematical Territory
- Teachers' Mathematical Work



PCK

- Pedagogical Content Knowledge is primarily described as some melding of Mathematical and Pedagogical Knowledge - a subject-specific pedagogy.





Integrated Knowledge

- Mathematical/Pedagogical knowledge that is meaningfully integrated:
  - How does teacher education promote integration of mathematical and pedagogical knowledge?
  - How does teacher education elicit the mathematical and pedagogical work of teaching?

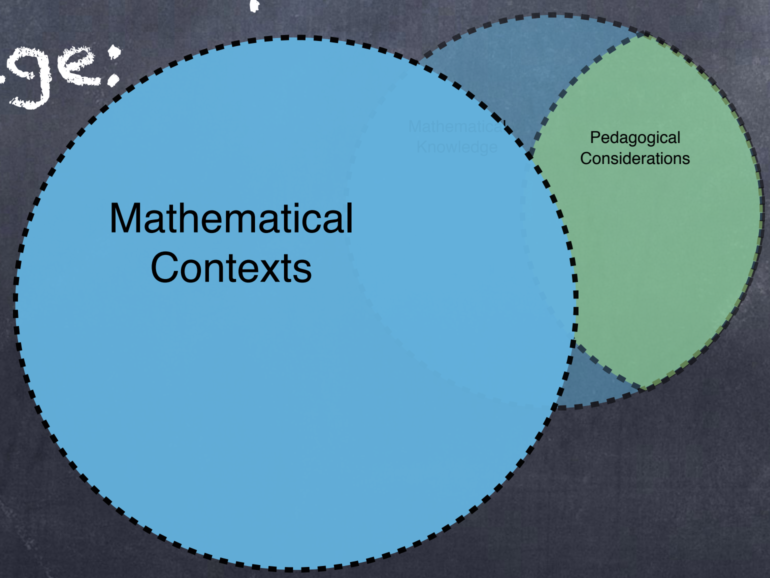
Tasks in Teacher Education

- According to Hiebert & Wearne (1993), "what students learn is largely defined by the tasks they are given" - which applies to prospective teachers as well

- Tasks become the mediating tool for teachers' learning; the quality of instruction and the success of the tasks depends on whether they unfold in ways that allow prospective teachers to learn through them (e.g., Kilpatrick et al, 2001)

Prominent Perspective

- There seems to be a prominent perspective on tasks in teacher education designed that promote developing such integrated mathematical/pedagogical knowledge:



Mathematical Contexts

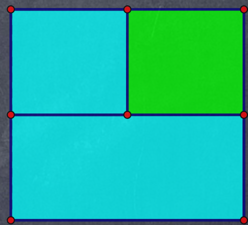
FRACTIONS

(Mathematical Context)

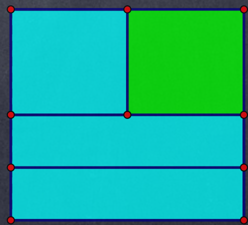
Pedagogical considerations

Common Student Difficulties, e.g.:
- Equipartitioning
- Operations
- Equivalence
- Comparing

- Tasks in teacher education may look like:
  - One student says that the fraction (shaded blue) is 2/3. How might you as the teacher respond?

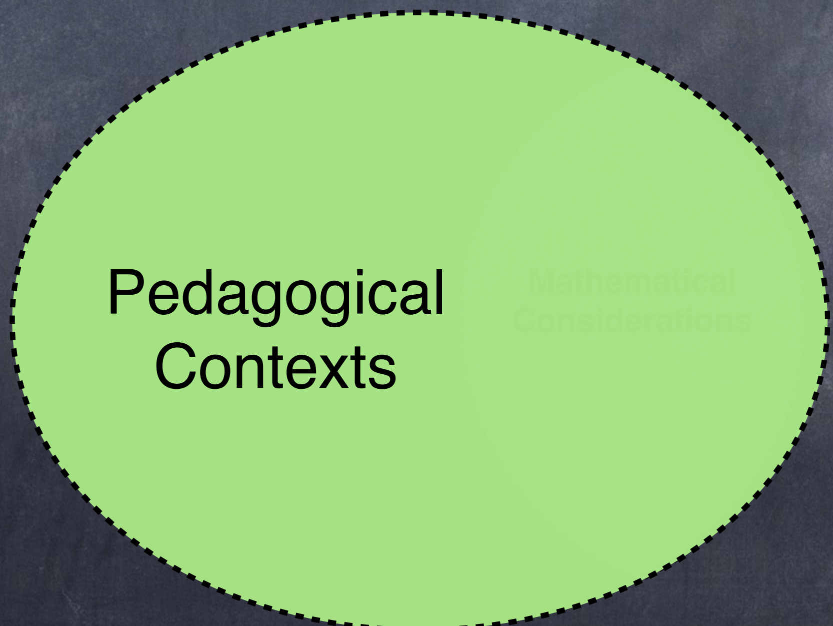


  - Another student says it is 3/4, by drawing the following picture. Is the student correct? Explain. How would you justify this to the class?



Pedagogical Contexts

- What might tasks in teacher education look like that develop integrated mathematical/pedagogical knowledge from the flipped perspective?



I. A Parallelogram Task

Pedagogical Context

- Recently, you introduced your class to the area formula for parallelograms, A=bh. You justified the formula by removing and relocating a triangle, as below.

Pedagogical Context

- Students' conceptual understanding about area remains somewhat fragile and needs to be continually associated with enumerating square units
- Students have difficulty transferring "remove-relocate" argument to unusually tall parallelograms (Wertheimer)

- Students have difficulty understanding that either length or width can serve as the base.

Discussion

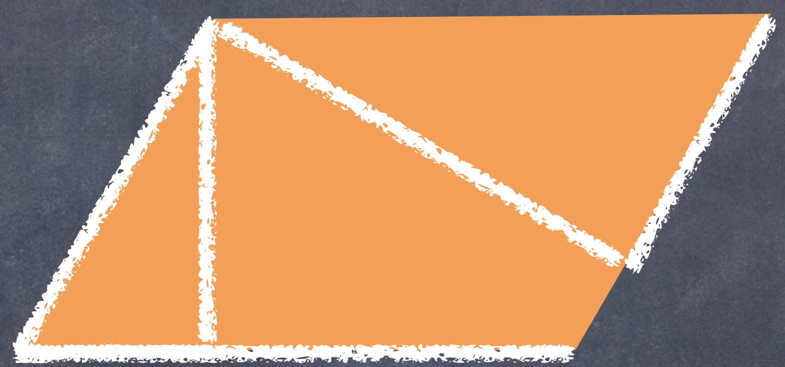
- How might you approach preparing a lesson with these pedagogical considerations? Think about a task for students in a subsequent lesson.
  - What might be important?
  - What examples of parallelograms might you look at?
  - What concepts might you emphasize?

Bill's Parallelogram

- Bill decides that he is going to model the Leaning Tower of Pisa with a parallelogram. However, he is having difficulty identifying dimensions to use. He would like every dimension (length, width, and both heights) to be integer values - in addition, he would like the height(s) to "split" the base(s) at an integer value.

Bill's Parallelogram

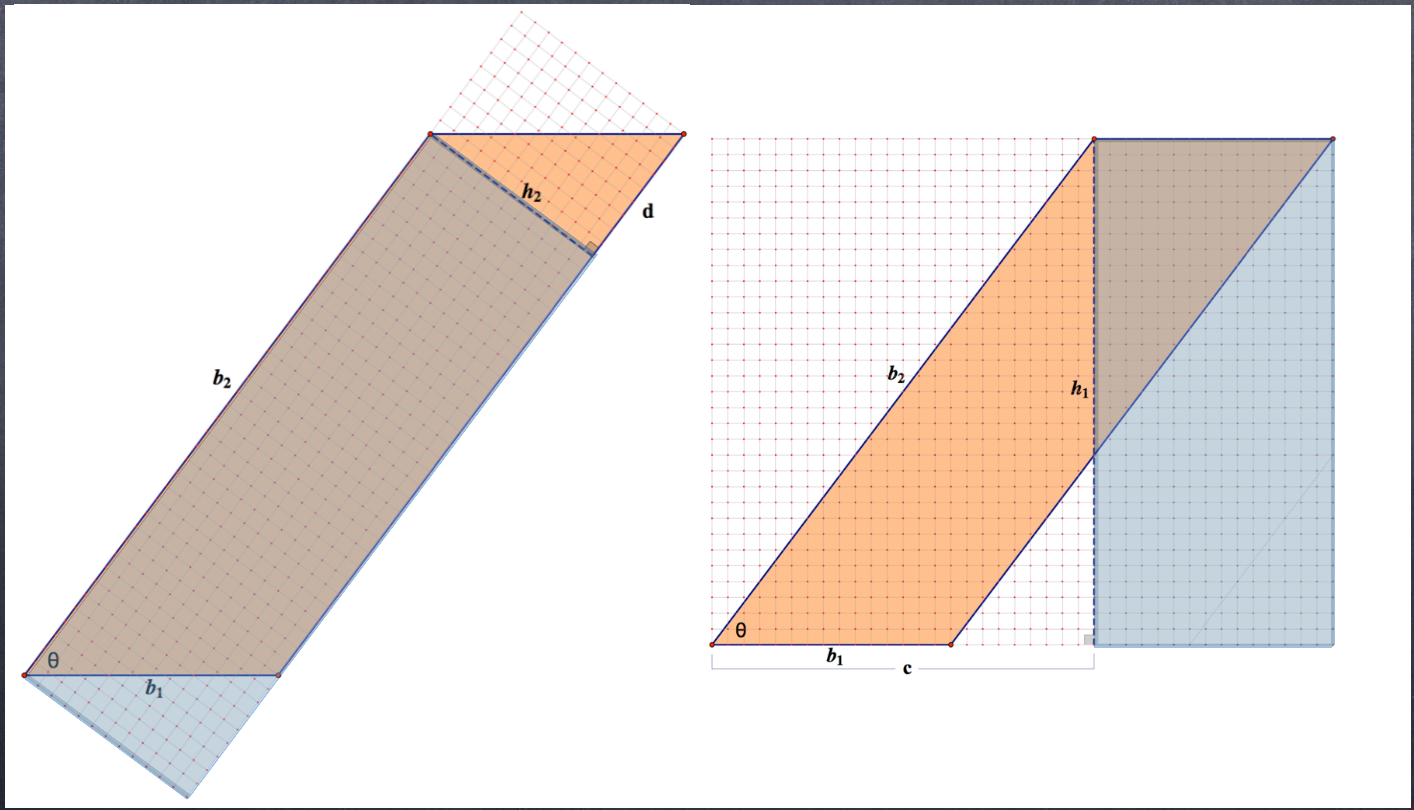
- In other words, Bill wants b1, b2, h1, h2, c, and d all to be integer values.



- Find values of b1, b2, h1, and h2 that meet this purpose - make sure they are also plausible for the Leaning Tower of Pisa. (What's θ?)

Bill's Purpose

- What might have been Bill's purpose for wanting a parallelogram with these constraints?

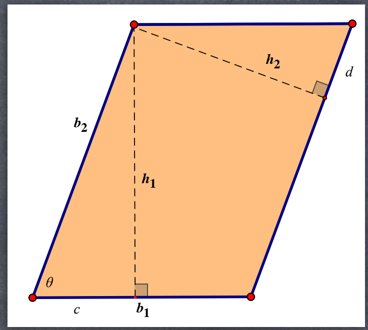


Mathematical Work

- In general, what constraints on b1, b2, and θ (assume θ is the acute angle) result in such classes of parallelograms?
  - b1, b2, h1, h2 are integers
  - b1, b2, h1, h2, c, d are integers

Mathematical Work

- $\sin\theta = h1/b2 = h2/b1$; Since these are integer values, $\sin\theta$ is a rational number (between 0 and 1), which means there exist relatively prime m and n (with $m<n$) s.t. $\sin\theta = m/n$.

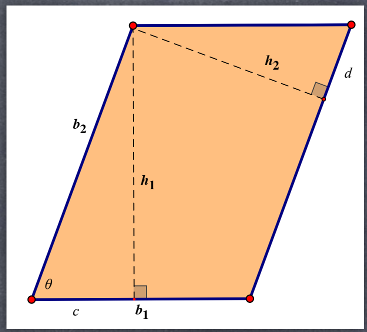


- Since $h1=b2(m/n)$ and $h2=b1(m/n)$ are also integers, then n must divide both b1 and b2.

- Therefore, for b1, b2, h1, h2 to be integers, b1 and b2 must have a common divisor, n, and the acute angle must be such that $\sin\theta = m/n$ (for $0<m<n$).

Mathematical Work

- In order for c and d to also be integers, the value for n must be the hypotenuse length of a primitive pythagorean triple (e.g., 5, 12, 13). And the value for m must be one of the other two values in the triple (e.g., 5, 12, 13).



- Thus, to construct a "Bill's Parallelogram":
  - Pick a value for n, the hypotenuse of a primitive pythagorean triple; select a value for m, one of the other two values in the triple
  - Select two multiples of n to be b1 and b2
  - Select an angle so that $\sin\theta = m/n$.

n=13
m=12

b1=65
b2=195

A Parallelogram Task

- This example used pedagogical contexts in teacher education tasks (such as the one described) as a means to explore mathematical considerations
- Such tasks in teacher education help reveal the mathematical work of teaching, in ways that promote integrated mathematical/pedagogical knowledge

II. Uses of Mathematical Knowledge in teaching

Uses of Knowledge

McCrory, et al., 2012

- Decompressing
  - Unpacking a topic's mathematical complexity in order to make it comprehensible
- Trimming
  - Removing complexity while maintaining mathematical integrity
- Bridging
  - Making connections across topics, assignments, representations, and domains

Uses of Knowledge

- Decompressing/Unpacking: Working with students' knowledge as it grows necessitates deconstructing teachers' own mathematical knowledge into less polished form, where elemental components are visible.
  - 1005 as one-hundred five
  - solving equations to allow interpretation of the results 0=0 or 0=3
- Complexities are in a local (micro-level) neighborhood of the context being taught

Uses of Knowledge

- Trimming: Any concept can be taught in some intellectually honest way; carefully attending to ideas in a perhaps overly-polished form that removes or hides complexities with the intent of simplifying the concept:
  - bad example: "multiplying makes bigger"
  - slope of constant rate of change of linear functions in light of the ways it emerges as instantaneous in calculus
- Complexities are in a broader (macro-level) neighborhood of the context being taught

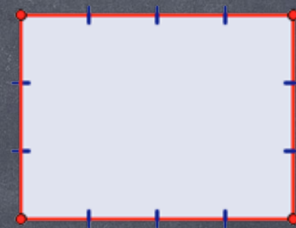
Micro-level Trimming

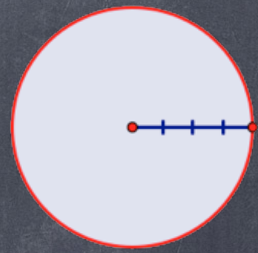
- Bill's Parallelogram

  - Mathematical issues related to area formulas, irrational lengths, piecing together partial squares, etc., are micro-level (local) complexities
  - Intent was to remove/hide these complexities in order to further emphasize and make desired ideas clear, undistracted by aspects that may unnecessarily complicate

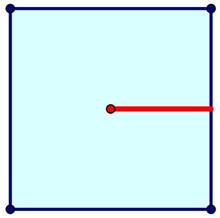
Macro-level Decompressing

- Square Perimeter
  - Perimeter as the sum of all sides is sufficient for polygons; but moving into circles complicates this idea in two ways: 1) multiplicative reasoning (not additive); 2) indirect measurement
  - Dave developed the following task when discussing the perimeter of a square in light of this





How much longer is the perimeter of the square compared to the "middle" length pictured?



Macro-level Decompressing

- Square Perimeter
  - Mathematical issues related to perimeter, multiplicative reasoning (not additive), indirect measurement, etc., are macro-level (local) complexities, e.g., circles
  - Purposefully introduced (not removed/ hidden) these complexities in order to prepare and unpack ideas for future developments of the concept, in ways intended to help students' transition

Navigating Complexities

- Conceptualizing Micro- and Macro-levels of Trimming and Decompressing in relation to the neighborhood of the mathematical complexities

Response to complexity / Neighborhood of complexity	Trimming (Removing complexity)	Decompressing (Unpacking/Highlighting complexity)
Micro-level (Local neighborhood) (To make ideas comprehensible)	Micro-level Trimming	Micro-level Decompressing
Macro-level (Distant neighborhood) (To maintain integrity)	Macro-level Trimming	Macro-level Decompressing

Implications

- Part I: Tasks in teacher education that use pedagogical contexts to explore mathematical considerations promote integrated mathematical/pedagogical knowledge and reveal mathematical work of teaching

- Part II: 2x2 Framework for navigating mathematical complexities addresses both mathematical nature (local or distant neighborhood) and pedagogical responses (trimming or decompressing), and may be useful tool for teacher education

Questions? Comments?

Thanks!

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