The Dilemma of Advanced Mathematics: Instructional Approaches for Secondary Mathematics Teacher Education

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20 November 2016

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A Dilemma

- Content knowledge for secondary teachers…

A full range of **advanced mathematics** courses (BREADTH)

- Advanced mathematics underpins the content of secondary mathematics (e.g., MET II, 2012)

Yet…

- Taking more advanced mathematics courses does not necessarily improve instruction (e.g., Monk, 1994)
- Teachers do not value such courses (Zazkis & Leikin, 2010)

Focus courses explicitly on the **content they will teach** (DEPTH)

- Knowing what one teaches in deep and profound ways leads to more flexible instruction (e.g., Brown & Borko, 1992)

Yet…

- Knowing nothing beyond what one teaches is imprudent in any field; mathematicians and mathematics educators support knowing content beyond what one teaches (e.g., Ball, Thames, Phelps, 2008; McCrory, et al, 2012)

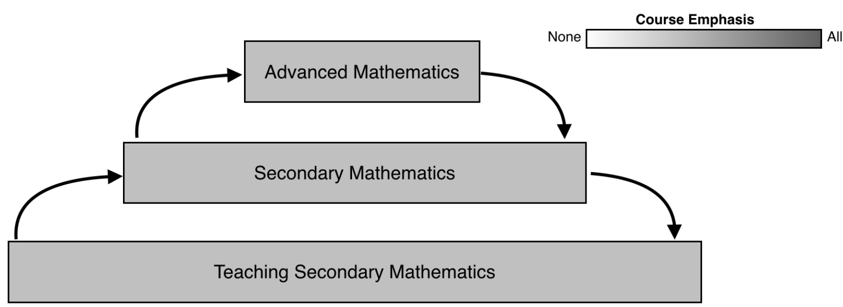
In the Direction of a Solution

- How might we approach *instruction* in advanced mathematics courses, with the intent of being *valuable preparation for secondary teachers*?

  1. An **instructional model** that connects advanced mathematics not just to the *content* of secondary mathematics but to the *teaching* of secondary mathematics

  2. **Instruction that** also **models** good teaching of mathematics.

Part I:
An Instructional Model



Introduction

$\frac{21}{\;\;7\;\;}$ ... ___

(Q, +, x)

$\left(\;\; 3 \;\; \right) \left(\; \ldots + 2x \right) =$ ____

(R[x], +, x)

For $f(x) = \frac{x+1}{x-3}, g(x) = \frac{3x+1}{x-1},$ ... ___

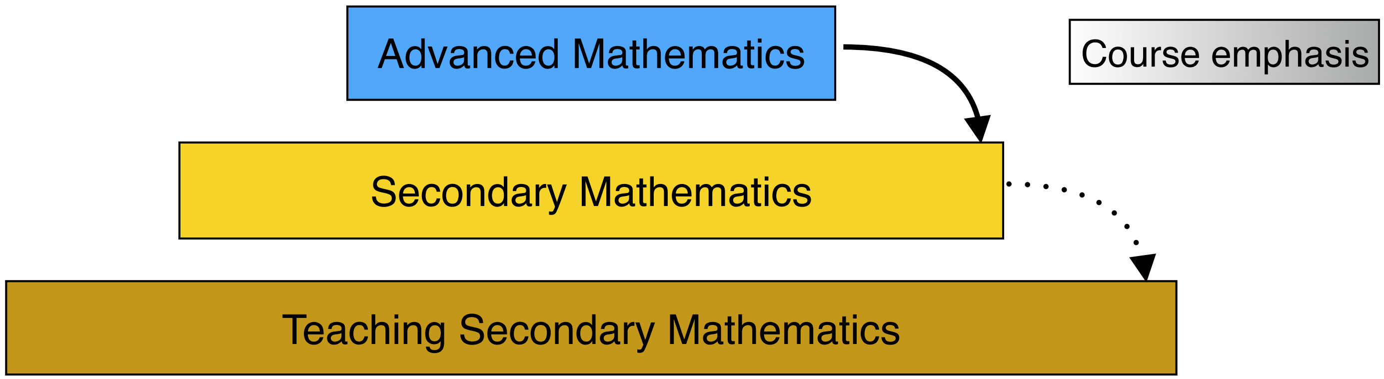
(F[x], ∘)

The objects of interest are *mathematically related.*
But does that mean knowing these connections to abstract algebra is *pedagogically meaningful?*

Instructional Model

Traditional model

(if anything) "Make connections to school mathematics"

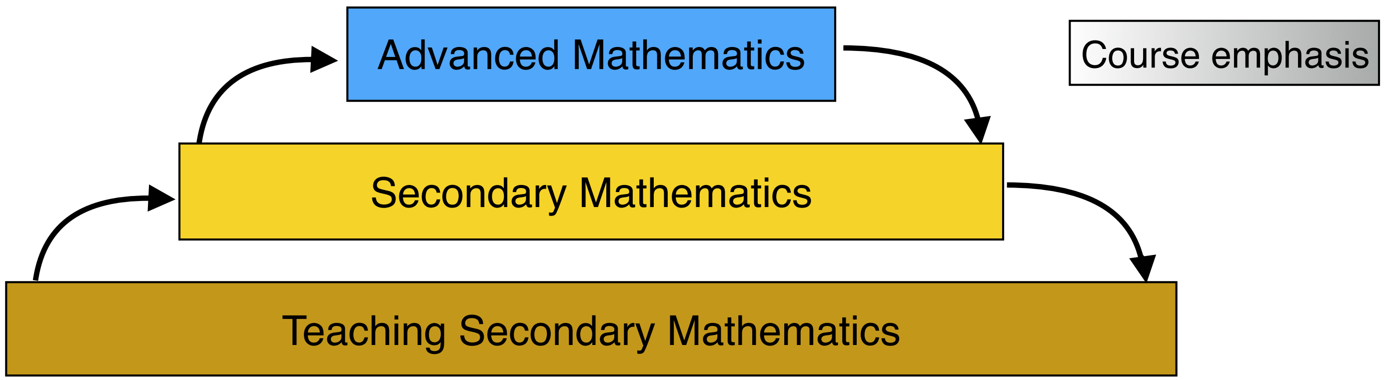


Trickle down effect: implicit hope is that a byproduct of learning advanced mathematics will be responding differently to instructional situations in the future.

Instructional Model

Alternate model

'Build up from' and 'Step down to' Teaching Practice



Begin with realistic situations in teaching secondary mathematics, where doing some facet of instruction well is well-situated to being learned in advanced mathematics.

Two comments

1. Presume “Advanced mathematics” to be true to its rigorous form and reasoning (not a ‘watered-down’ version)
   - Mathematical goal must include rigorous content
2. Intended pedagogical situations must be authentic as well as responses improved in some meaningful way by learning the advanced mathematics.
   - Pedagogical goal must be clear and connected to some aspect of good mathematics instruction

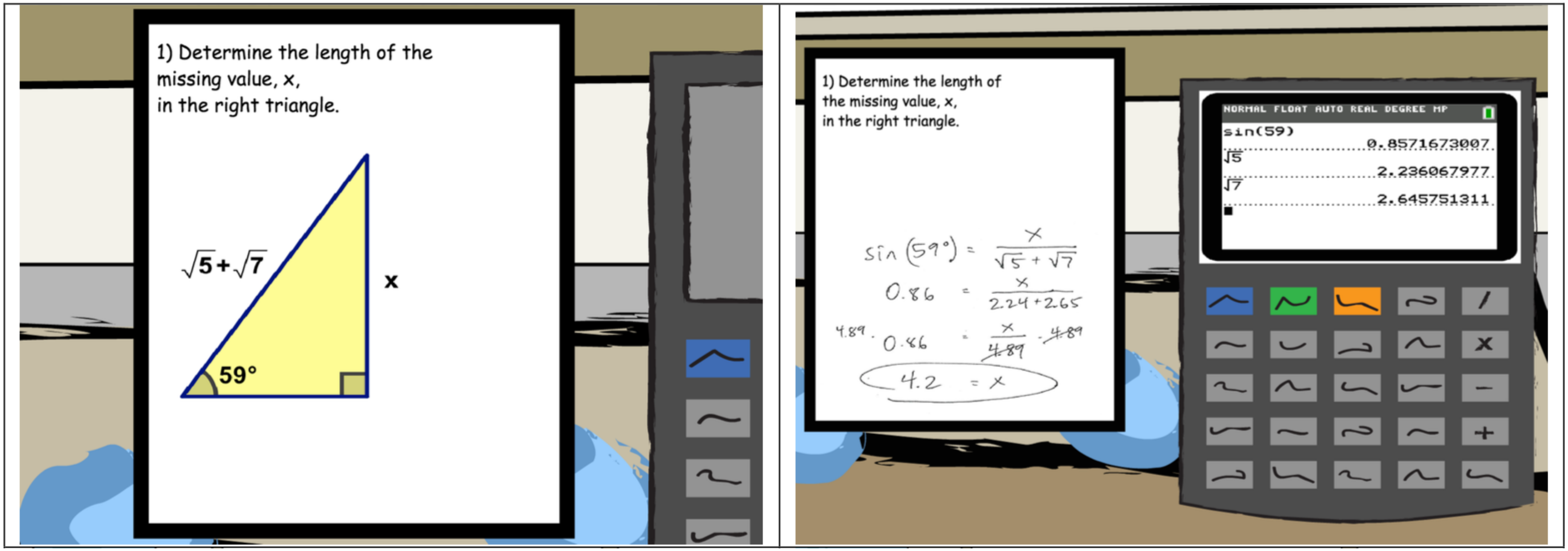
An Example of an **Instructional Module** from Real Analysis

Example Module

- Mathematical goal: Proofs of the Algebraic Limit Theorems for Sequences.
  - If $(a_n) \to a$, and $(b_n) \to b$, then:
    - i) $(ca_n) \to (ca)$, *for all c in R*; (Scalar property)
    - ii) **$(a_n+b_n) \to (a+b)$**; **(Sum property)**
    - iii) **$(a_nb_n) \to (ab)$**; **(Product property)**
    - iv) $(a_n/b_n) \to (a/b)$, *provided b is not 0*. (Quotient)
- Pedagogical goal: In relation to rounding, to be able to ***clarify mathematical limitations*** *in students' statements or arguments*, and to ***select examples*** that exemplify nuances within and around an idea

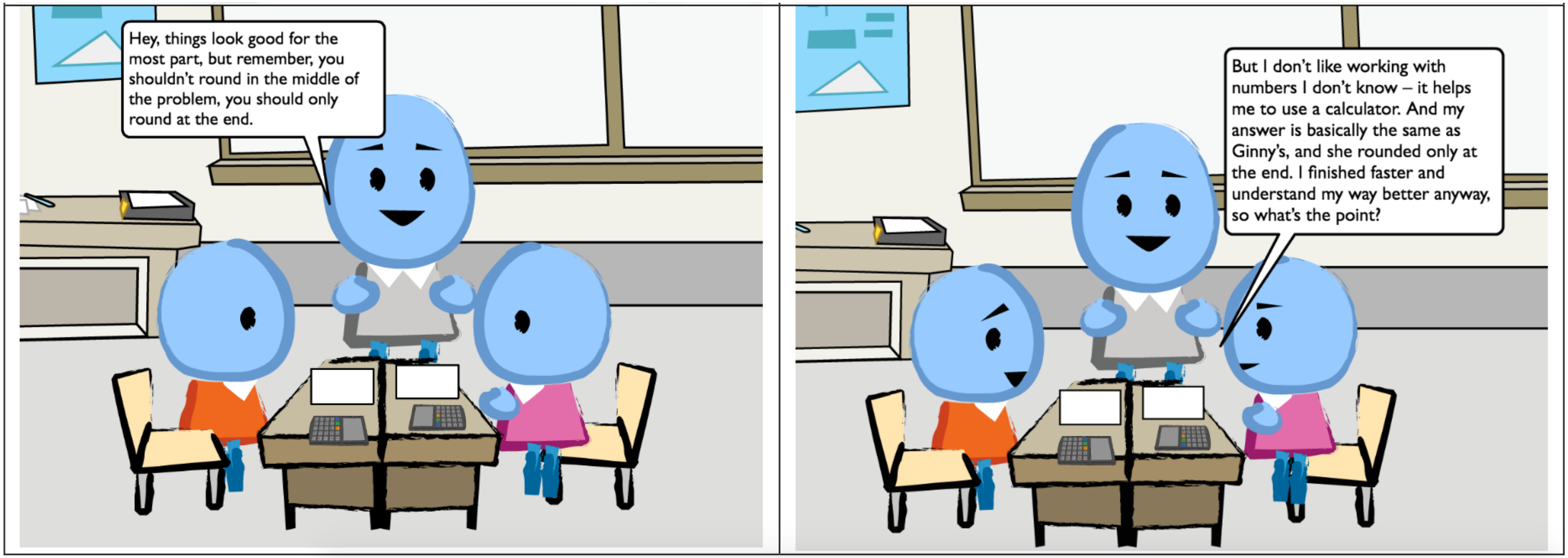
‘Building Up’

- An instructional situation:



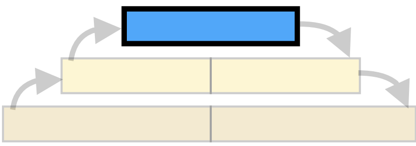
‘Building Up’

- An instructional situation:



Real Analysis

- Introductory remarks for considering real analysis:
  - A rounded (or truncated) number is an approximation for a number.
    - For some theoretical real number, a, then a rounded number, a_{appr}, is an approximation of this real number where the difference - or error - is relatively small (and bounded): i.e., $| a_{appr} - a | < e$
  - This brings to mind convergent sequences… "A sequence (a_n) converges to a real number a if for all *epsilon*>0, there exists a natural number N s.t. for all $n \geq N$, $| a_n - a | <$ *epsilon*."



Proof 1

If $a_n \to a$, and $b_n \to b$, then:

ii. $a_n + b_n \to a + b$

Proof. Let $\varepsilon > 0$. For all n, $|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$,

so $|(a_n + b_n) - (a+b)| \leq |a_n - a| + |b_n - b|$.

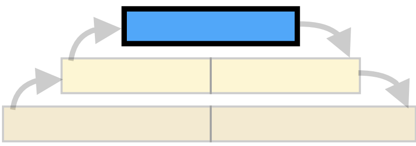
Since $a_n \to a$, there exists an N_1 such that for all $n \geq N_1$, $|a_n - a| < \frac{\varepsilon}{2}$.

Since $b_n \to b$, there exists an N_2 such that for all $n \geq N_2$, $|b_n - b| < \frac{\varepsilon}{2}$.

Therefore, for $n \geq \max[N_1, N_2]$, $|(a_n + b_n) - (a+b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$.

So for any $\varepsilon > 0$, the sequence $(a_n + b_n)$ is within ε of $(a+b)$ (for $n \geq \max[N_1, N_2]$). QED.

If we approximate π by 3.1 and 1/3 by 0.33, then 3.1+0.33=3.43 is within .1+.01=0.11 of π+1/3



Proof 2

If $a_n \to a$, and $b_n \to b$, then:

iii. $a_n b_n \to ab$

Proof. Let $\varepsilon > 0$. For all n, $|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |a_n b_n - ab_n| + |ab_n - ab|$,

so $|a_n b_n - ab| \leq |b_n||a_n - a| + |a||b_n - b|$.

Since $b_n \to b$, there exists an N_2 such that for all $n \geq N_2$, $|b_n - b| < \frac{\varepsilon}{2|a|}$.

Since every convergent sequence is bounded, there is an M such that for all n, $|b_n| \leq M$.

Since $a_n \to a$, there exists an N_1 such that for all $n \geq N_1$, $|a_n - a| < \frac{\varepsilon}{2|M|}$.

Therefore, for $n \geq \max[N_1, N_2]$, $|a_n b_n - ab| \leq |M||a_n - a| + |a||b_n - b| < |M|\frac{\varepsilon}{2|M|} + |a|\frac{\varepsilon}{2|a|} = \varepsilon$.

So for any $\varepsilon > 0$, the sequence $(a_n b_n)$ is within ε of ab (for $n \geq \max[N_1, N_2]$). QED.

If we approximate π by 3.1 and 1/3 by 0.3, then 3.1x0.3=0.93 is within .1(π+1/3)≈0.3475 of (1/3)π

‘Stepping Down’

- Returning to the instructional situation:

Can you ***craft an example*** *- i.e., modify the problem - to help* **exemplify / show** *the student some of the potential issues with their approach?*

Initial Problem

$$\sin(59^\circ) = \frac{x}{\sqrt{5}+\sqrt{7}} \quad \leftarrow \approx 4.88$$

$$0.85 \times 4.88 \approx x$$

Original .01 error grows *up to* .01(~5.74)≈.0574

Modified Problem

$$\sin(59^\circ) = \frac{x}{360\sqrt{5}} \quad \leftarrow \approx 804.98$$

$$0.85 \times 804.98 \approx x$$

Original .01 error grows *up to* .01(~805.84)≈8.0584

‘Stepping Down’

- How much error?

$$\frac{2}{7}x - 600 = \frac{6}{11}$$

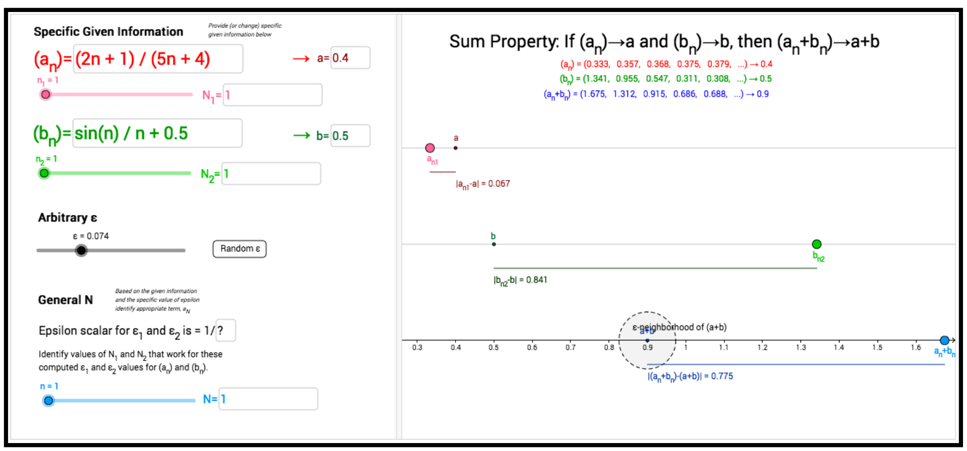
$$0.28x = 600.54$$

$$x = 2{,}144.78$$

Conclusion

- Mathematical goal: Connecting sequences and proofs of algebraic limit theorems to operating on rounded numbers provides a **meaningful context with which to teach these proofs to students**
- Pedagogical goal: Making explicit the connection to *crafting examples* and instructional tasks, by exploring ways in which one can manipulate this kind of error, provides **an application of these theorem proofs for use in secondary mathematics teaching**.

Part II: Modeling Instruction



Modeling instruction

- An age-old adage about teachers: “We teach how we were taught.”
- If we provide opportunities for teachers to *learn* mathematics in ways that model good instruction, this sets them up to *teach* in ways that leverage such good instructional practices
- In this section, I only address one such instructional approach that makes use of dynamic technology

Dynamic Technology

- Dynamic technologies provide an ability to *create* and *manipulate* objects such that:
  1. manipulation is direct (i.e., a user points and drags)
  2. motion is continuous (i.e., change happens in real time)
  3. the environment is immersive (i.e., the interface is minimally intrusive)

Three Comments

1. An advanced mathematics course is an opportunity for PSTs, as students, to experience learning mathematics through the use of a dynamic technology (it is not an opportunity to teach PSTs about features of a particular technology)

2. Again, the context is “advanced” mathematics - the use of technology should be befitting for the content to be learned (i.e., proof, etc.)

3. Specifically, I presume technology is most powerful when students (not instructors) are intended to interact with it.

An Example of a **Dynamic Proof Visualization** from Real Analysis

Dynamic Proof Visualizations

- On the **“dynamic”** aspects:
  - Dynamic proof visualizations make use of the “interactive” environment and nature of dynamic technologies.
  - Real analysis, in particular, takes things we often think of as “static” (i.e., real numbers and real-valued functions) and makes them “dynamic” (i.e., infinite sequences, approaching arbitrarily small epsilon-neighborhoods) - so using dynamic software might be especially productive.
    - However, unlike a point on a plane (where one can “drag” through the space of possible locations), “dragging” through the space of “sequences” or “functions” is not feasible

Dynamic Proof Visualizations

- On the **"proof"** aspects:
  - The use of dynamic technology is powerful in that it links the *general* claim being made to the (various) *specific* examples that both instantiate and substantiate (or refute) the claim.
  - Dynamic proof visualizations were not created to just convince students about the truth or falsehood of a *claim*, but rather to mimic some of the proof's processes, arguments, etc., to foster insight about the *proof of a claim.*
    - Notably, the "givens" of a claim are explicit; changing these allows students to explore why and in what ways the argument may fail with or without the given constraints.

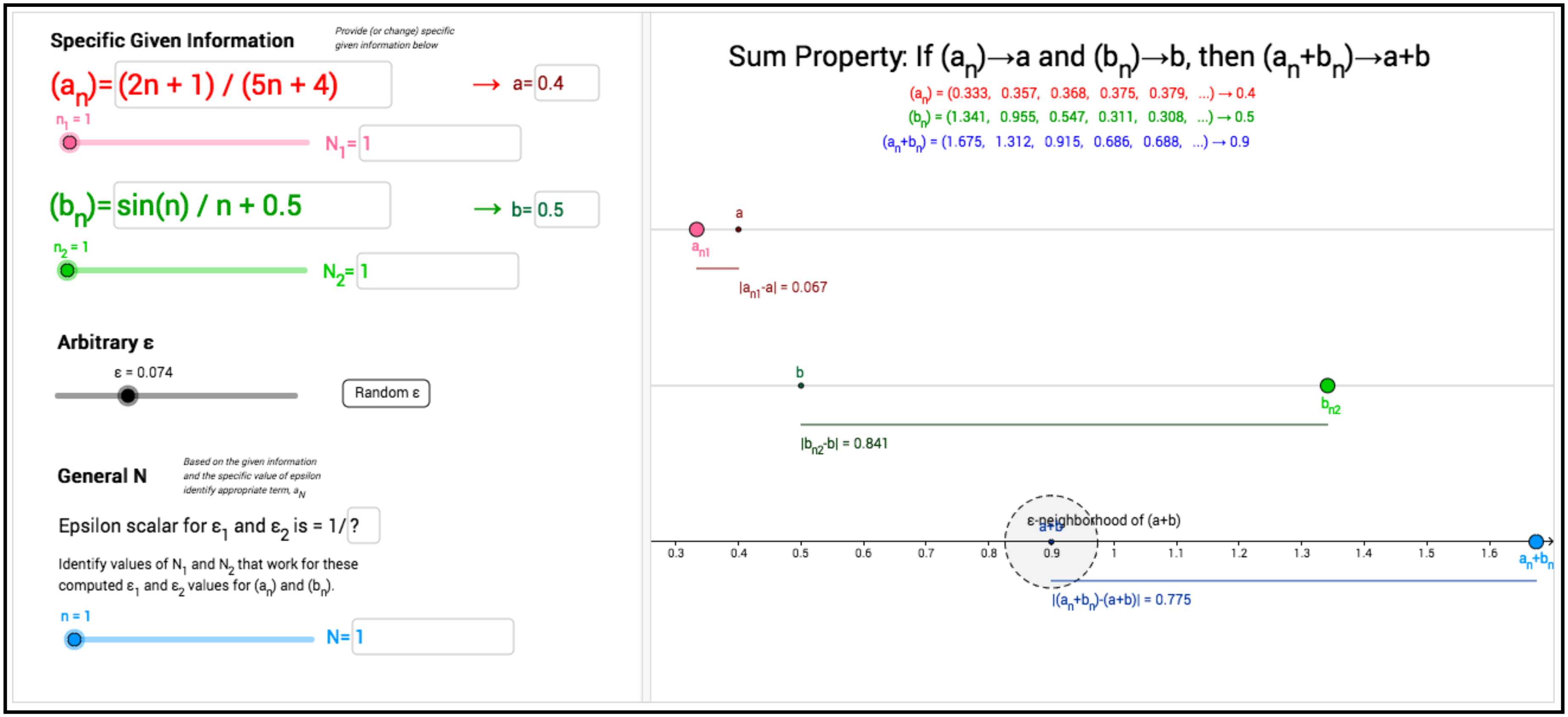
Dynamic Proof Visualizations

- On the **"visualization"** aspects:
  - Dynamic proof visualizations intend to make visual (and dynamic) something that is abstract (and static) - in this sense, the visuals aim to provide mathematical insight
  - However, reliance on "visuals" can often also be a barrier to proof (i.e., pictures can often made additional assumptions about the given context… some functions cannot actually be drawn)
  - In addition, technology is discrete - in a course about real numbers and infinite processes, there will always be a moment at which point things break down

Algebraic Limit Theorems

- We will consider an example regarding a proof previously discussed:
  - Theorem. If $(a_n) \rightarrow a$, and $(b_n) \rightarrow b$, then $(a_n+b_n) \rightarrow (a+b)$
  - Definition: A sequence (a_n) converges to a real number a if for all *epsilon*>0, there exists a natural number N s.t. for all $n \geq N$, $| a_n - a | <$ *epsilon*.

Algebraic Limit Theorems



Modeling instruction

- Specifically, the dynamic proof visualization:
  - Allowed an ability to dynamically interact with the given sequences in the claim.
  - It helped clarify some of the various *epsilons* and *Ns* referenced in the proof, as well as their uses and coordination throughout.
- Generally, the dynamic proof visualization:
  - Modeled the utility of the software for exploring dynamic concepts and claims in mathematics
  - Modeled not just "instructor" but "student" use

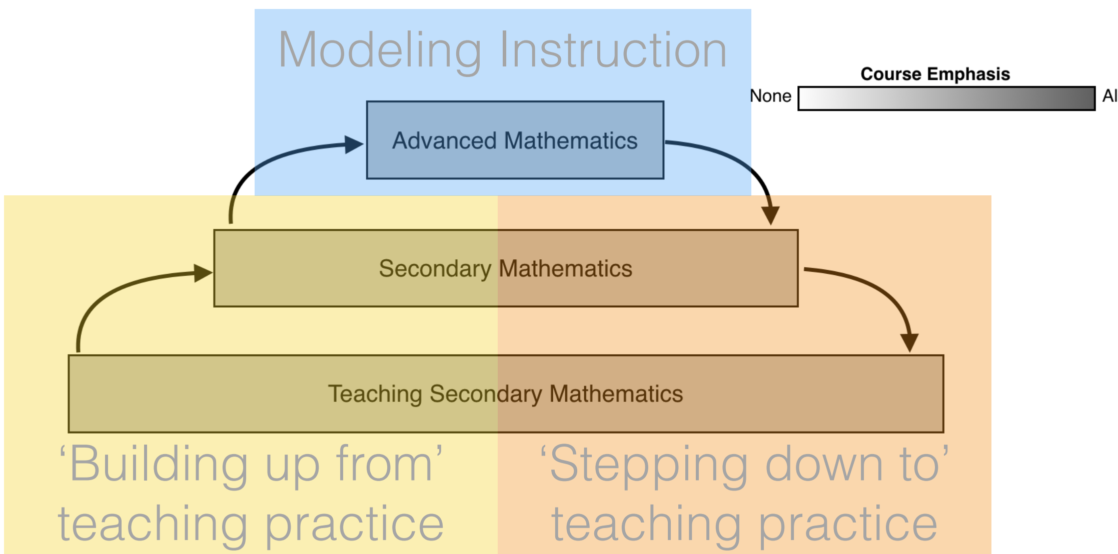
Conclusion

- I have discussed two potential solutions regarding the dilemma of advanced mathematics courses in secondary teacher education, that relate to productive instructional practices in such courses

1. **An instruction model of building up from and stepping down to teaching practice.**
    - Rather than simply connecting advanced mathematics to the *content* of secondary mathematics, tasks are designed to illuminate applications for the *teaching* of secondary mathematics.
2. **Modeling good instruction via dynamic software**
    - Dynamic proof visualizations modeled use of dynamic software: i) utilize dynamic features to help visualize dynamic ideas and claims; and ii) allow students, not just instructors, to interact with the technological environment.

Conclusion

- These two instructional approaches can complement each other, focusing on different parts of advanced mathematics instruction, and can both be used to make conversations about and connections to teaching explicit.



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Thanks!

Questions? Comments?